ON THE STABILITY IN THE CRITICAL CASE OF A ZERO ROOT FOR SYSTEMS WITH TIME LAG

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A practical method is given for solving the problem on the undisturbed motion of a system with time lag for the critical case when one of the roots is zero.

1. Statement of the problem. We consider the equation

$$\frac{dx_i(t)}{dt} = \sum_{j=1}^{n} \int_{-\tau}^{0} x_j(t+\vartheta)\, d\eta_{ij}(\vartheta) + X_i(x_1(t+\vartheta), \ldots, x_n(t+\vartheta)) \qquad (i=1, \ldots, n) \tag{1.1}$$

where the integrals are taken in the sense of Stieltjes [1,2]; the $X_i(x_1(\vartheta), \ldots, x_n(\vartheta))$ are functionals defined for sectionally continuous functions $x_i(\vartheta)$ of the argument $\vartheta(-\tau < \vartheta < 0)$ and represent nonlinear perturbations.

More precisely, the X_i satisfy the Lipschitz condition in the x_i

$$|X_i(x_1''(\vartheta), \ldots, x_n''(\vartheta)) - X_i(x_1'(\vartheta), \ldots, x_n'(\vartheta))| \leqslant L\|x'' - x'\| \tag{1.2}$$

$$\|x(\vartheta)\| = \sup\{|x_1(\vartheta)|, \ldots, |x_n(\vartheta)|\}, \qquad L = L_1\{\|x''\| + \|x'\|\}^{\alpha_1} \quad \text{при } -\tau \leqslant \vartheta \leqslant 0 \tag{1.3}$$

Here L_1 and α_1 are positive numbers.

The condition (1.3) together with (1.2) guarantees the nonlinearity of the terms X_i in Equations (1.1). Obviously, $X(0, \ldots, 0) \equiv 0$. The motion $x \equiv 0$ will be called the undisturbed motion of the system (1.1). We shall assume that if one substitutes in X_i a function $x(y, \vartheta)$ which is analytic in y then one obtains an analytic function of y.

From Equation (1.1), one can obtain equations with lag under special assumptions on the Stieltjes measure $d\eta_{ij}(\vartheta)$. Thus, for example, let us assume

$$d\eta_{ij}(\vartheta) = 0 \quad \text{при } \vartheta \neq 0, \quad \vartheta \neq -\tau$$

$$d\eta_{ij}(0) = a_{ij}, \quad d\eta_{ij}(-\tau) = b_{ij} \quad (a_{ij}, b_{ij} = \text{const})$$

$$X_j(x_1(t+\vartheta), \ldots, x_n(t+\vartheta)) = F_i(x_1(t), \ldots, x_n(t), x_1(t-\tau), \ldots, x_n(t-\tau))$$

where F is an analytic function of its arguments. In this case we obtain the following system of equations for the disturbed motion with lag [delay]

$$\frac{dx_i(t)}{dt} = \sum_{j=1}^{n} a_{ij}x_j(t) + \sum_{j=1}^{n} b_{ij}x_j(t-\tau) + F_i(x_1(t), \ldots, x_n(t-\tau)) \qquad (i = 1, \ldots, n) \tag{1.4}$$

The system of linear equations with lag

$$\frac{dx_i(t)}{dt} = \sum_{j=1}^{n} \int_{-\tau}^{0} x_j(t+\vartheta)\, d\eta_{ij}(\vartheta) \tag{1.5}$$

we shall call the system of the first approximation.

Let us consider the characteristic equation of the first approximation

$$\triangle(\lambda) \equiv \left| -\delta_{ij}\lambda + \int_{-\tau}^{0} e^{\lambda\vartheta}\, d\eta_{ij}(\vartheta) \right| = 0 \tag{1.6}$$

Let us suppose that among the denumerable set of roots $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ of Equation (1.6) there is one zero root ($\lambda_1 \equiv 0$), while each of the remaining roots has a negative real part

$$\operatorname{Re}\lambda_j \leqslant -2\alpha \qquad (j > 1) \tag{1.7}$$

In this case we have the so-called critical case of one zero root. This problem was solved by Liapunov [3, 4] for motions described by ordinary differential equations. In this case the stability of the undisturbed motion of the system of the first approximation does not imply the stability of the undisturbed motion of the entire system. The nonlinear terms have a definite effect on the stability or instability of the motion. A number of other critical cases for ordinary differential equations were investigated in the works of Liapunov, Chetaev, Malkin, Kamenkov, Krasovskii and others. For systems with lag the corresponding problem has not yet been considered in the general case.

For systems with lag, Bellman [7] has shown that if the undisturbed motion of the first approximation is asymptotically stable ($\operatorname{Re}\lambda_j = -2\alpha$), then the undisturbed motion of the entire system is asymptotically stable.

Here we consider the stability of the undisturbed motion $x = 0$ for the system (1.1) with lag in the critical case of one zero root.

In order to determine $dx_i(t)/dt$ at a given time it is necessary to know $x_i(t)$ not only at the given moment of time t, but also over the interval $(t - r, t)$ of length r. It is, therefore, convenient to select [1] for an element of the trajectory not the point $x_i(t)$ but the segment $x_j(t + \vartheta)$ $(-r \leqslant \vartheta \leqslant 0)$ of the trajectory. Hereby, one can treat this segment of the trajectory as a point in the function space B. Krasovskii [1] has shown that in a function space the equations with lag are replaced by a system of "ordinary" differential equations whose right-hand sides involve operators.

Let $x(t)$ be a solution of (1.1). A solution element in the function space will be given by $x_i(t + \vartheta)$ $(-r < \vartheta < 0)$. Corresponding to Equation (1.1) we will have an equivalent system of "ordinary" differential equations

$$\frac{dx_t(\vartheta)}{dt} = Ax_t(\vartheta) + R(x_t(\vartheta)) \tag{1.8}$$

where

$$x_t(\vartheta) = \{x_1(t + \vartheta), \ldots, x_n(t + \vartheta)\} = \{x_{1t}(\vartheta), \ldots, x_{nt}(\vartheta)\}$$

$$y(\vartheta) = Ax(\vartheta) = \begin{cases} \dfrac{dx_k(\vartheta)}{d\vartheta} & (-\tau \leqslant \vartheta < 0) \\ \sum\limits_{j=1}^{n} \int\limits_{-\tau}^{0} x_j(\vartheta)\, d\eta_{kj}(\vartheta) & (\vartheta = 0) \end{cases} \qquad (k=1,\ldots,n) \tag{1.9}$$

$$R(x_t(\vartheta)) = \begin{cases} 0 & (-\tau \leqslant \vartheta < 0) \\ X_k(x_{1t}(\vartheta), \ldots, x_{nt}(\vartheta)) & (\vartheta = 0) \end{cases} \qquad (k=1,\ldots,n) \tag{1.10}$$

It is obvious that the problem on the stability of the undisturbed motion of the system (1.8) $x_t(\vartheta) = 0$ is equivalent to the corresponding problem for the system (1.1), for if $x_t(\vartheta)$ is any solution of the system (1.8), then $x_t(\vartheta) = x(t + \vartheta)$, where $x(t + \vartheta)$ is a solution element of the system (1.1).

If the initial function $x_0(\vartheta)$ is sectionally continuous, then the operator A will be defined only for $t \geqslant r \geqslant 0$ if the corresponding solution is differentiable. In view of this, we shall assume that the initial functions are differentiable and we shall consider the operators only when $t \geqslant 0$.

This last statement does not exclude initial functions which are sectionally continuous, because after the passage of the time r these functions will be replaced by a differentiable segment of the solution which can be taken as an initial function.

2. Properties of a linear operator. Since Equation (1.6) has a simple root equal to zero, the determinant $\Delta(0)$ will vanish:

$$\Delta(0) \equiv \left| \int_{-\tau}^{0} d\eta_{kj}(\vartheta) \right| = 0 \tag{2.1}$$

Let us denote by $\Delta_{kj}(\lambda)$ the algebraic cofactor of the element standing in the kth row and the jth column of the determinant $\Delta(\lambda)$. Then

$$\Delta'(0) \equiv \sum_{j=1}^{n} \left\{ -\Delta_{jj}(0) + \sum_{\sigma=1}^{n} \int_{-\tau}^{0} \vartheta \cdot d\eta_{j\sigma}(\vartheta)\, \Delta_{j\sigma}(0) \right\} \tag{2.2}$$

Since $\Delta'(0) \neq 0$, there exist two numbers k_1, l_1 such that $\Delta_{l_1 k_1}(0)$ is not zero. Let us consider the functional

$$f[x_t(\vartheta)] = \sum_{j=1}^{n} \Delta_{jk_1}(0) \left\{ -x_{jt}(0) + \sum_{l=1}^{n} \int_{-\tau}^{0} \left[\int_{0}^{\vartheta} x_{lt}(\xi)\, d\xi \right] d\eta_{jl}(\vartheta) \right\} \tag{2.3}$$

defined for the differentiable functions $x_t(\vartheta)$ $(-r \leqslant \vartheta \leqslant 0)$. One can easily verify the identity

$$f[Ax(\vartheta)] \equiv 0 \tag{2.4}$$

Let

$$b(\vartheta) = (b_1, \ldots, b_n) = (\Delta_{l_1 1}(0), \ldots, \Delta_{l_1 n}(0))\, d = \mathrm{const}(\vartheta) \tag{2.5}$$

where $d = [\Delta_{l_1 k_1}(0)\, \Delta'(0)]^{-1}$ is independent of ϑ. We shall consider $b(\vartheta)$ to be a vector function in the space $x(\vartheta)$ which is constant for all ϑ on the interval $-r < \vartheta < 0$. Then

$$f[b(\vartheta)] = 1, \qquad A(b(\vartheta)) = 0 \tag{2.6}$$

From the condition (2.4) it follows that $f[x_t(\vartheta)]$ is an integral (solution) of the system

$$\frac{dx_t(\vartheta)}{dt} = Ax_t(\vartheta) \tag{2.7}$$

The integral $f[x_t(\vartheta)]$ will be called a functional integral. Let $x_0(\vartheta)$ be an initial function for the solution $x_t(\vartheta)$ of the system (2.7). Then

$$f[x_t(\vartheta)] = f[x_0(\vartheta)] \qquad \text{при } t \geqslant 0 \tag{2.8}$$

Suppose the initial function $x_0(\vartheta)$ is such that

$$f[x_0(\vartheta)] = 0 \tag{2.9}$$

Then the corresponding solution $x_t(\vartheta)$ will decrease asymptotically as an exponential function with exponent α, i.e.

$$\|x_t(\vartheta)\| < K\|x_0(\vartheta)\|\exp(-\alpha t) \qquad (K = \text{const})$$

It is hereby assumed that the condition (1.7) is satisfied.

In view of (2.8) and (2.9), all solutions which decrease exponentially will lie in the function space $\{x(\vartheta)\}$ in the plane

$$f[x_t(\vartheta)] = 0 \tag{2.10}$$

The plane (2.10) will be called the L-plane.

We call attention to the fact that the operator A is defined for $t > 0$ if the initial functions are differentiable, and for $t > r$ if the initial functions are sectionally continuous. The functional $f[x_t(\vartheta)]$ will satisfy, therefore, condition (2.4) and its consequence (2.8) only for differentiable $x_0(\vartheta)$ when $t \geqslant 0$, and it will satisfy those conditions with $t > r$ when $x_0(\vartheta)$ is sectionally continuous (with discontinuities of the first kind).

Let $x(\vartheta)$ be an arbitrary element in the function space B in which the operator A is defined and in which the functional $f[x(\vartheta)]$ has the above-indicated properties. Every such element $x(\vartheta)$ can be broken up into two components as follows:

$$y = f[x(\vartheta)], \qquad x(\vartheta) = z(\vartheta) + b(\vartheta)\,y, \qquad b(\vartheta) = \text{const}\,(\vartheta) \tag{2.11}$$

(or more precisely, $x_k(\vartheta) = z_k(\vartheta) + b_k y$ $(k = 1, \ldots, n)$).

The following condition holds:

$$f[z(\vartheta)] = 0 \tag{2.12}$$

Indeed,

$$f[z(\vartheta)] = f[x(\vartheta)] - yf[b(\vartheta)] = f[x(\vartheta)] - f[x(\vartheta)]\cdot 1 \equiv 0$$

since $f[b] = 1$ in view of (2.6), and $y = f[x(\vartheta)]$ in consequence of (2.11).

Equation (2.12) determines the L-plane in the function space B. The vector-function $b(\vartheta) = (b_1, \ldots, b_n) = \text{const}, \vartheta \in [-r, 0]$ does not lie in the L-plane because $f[b] = 1$ by (2.6). The above-given decomposition of $x(\vartheta)$ represents, therefore, geometrically a breaking-up of the vector-function $x(\vartheta)$ into two terms $z(\vartheta)$ and yb, with $z(\vartheta)$ situated in the L-plane and with yb being colinear with the constant vector b.

The decomposition of the function $x(\vartheta)$ into two terms $z(\vartheta)$ and $yb(\vartheta)$ is unique. Indeed, suppose that there were two decompositions of $x(\vartheta)$ of the form

$$x(\vartheta) = z_1(\vartheta) + y_1 b = z_2(\vartheta) + y_2 b, \qquad \text{or} \qquad z_1(\vartheta) - z_2(\vartheta) = b(y_2 - y_1)$$

Let us assume that $y_1 \neq y_2$. Operating with f on both sides of the displayed equality we obtain

$$-(y_1 - y_2) f[b] \equiv y_2 - y_1 = f[z_1(\vartheta)] - f[z_2(\vartheta)] \equiv 0, \quad \text{or} \quad y_1 \equiv y_2, \quad z_1 \equiv z_2$$

It follows from this that if $x(\vartheta) = 0$, then $z(\vartheta) = y = 0$.

The system (2.7) of the first approximation takes on the following form in terms of $z_1(\vartheta)$ and $y(t)$:

$$\frac{dy}{dt} = 0, \qquad \frac{dz_t(\vartheta)}{dt} = Az_t(\vartheta), \qquad f[z_t(\vartheta)] = 0 \tag{2.13}$$

Let us express Equations (1.8) in terms of $z_t(\vartheta)$ and $y(t)$. We obtain

$$\frac{dy}{dt} = \frac{d}{dt} f[x_t(\vartheta)] = f\left[\frac{dx_t(\vartheta)}{dt}\right] = f[Ax_t(\vartheta)] + f[R(x_t(\vartheta))] = f[R(x_t(\vartheta))]$$

$$\frac{dz_t(\vartheta)}{dt} = \frac{dx_t(\vartheta)}{dt} - b\frac{dy}{dt} = Ax_t(\vartheta) + R(x_t(\vartheta)) - bf[R(x_t(\vartheta))] =$$
$$= Az_1(\vartheta) + yA(b) + R(z_t(\vartheta) + by) - bf[R(z_t(\vartheta) + by)]$$
$$A(b) = 0$$

Equations (1.8) can in this manner be reduced to the form

$$\frac{dy}{dt} = Y(y, z_t(\vartheta)), \qquad \frac{dz_t(\vartheta)}{dt} = Az_1(\vartheta) + Z(y, z_t(\vartheta), \vartheta) \tag{2.14}$$

Here Y is a functional defined by the formula

$$Y(y, z_t(\vartheta)) \equiv f[R(z_t(\vartheta) + by)] \equiv$$
$$\equiv -\sum_{j=1}^{n} \Delta_{jk_t}(0) X_j(z_{1t}(\vartheta) + b_1 y, \ldots, z_{nt}(\vartheta) + b_n y) \tag{2.15}$$

while $Z(y, z_t(\vartheta), \vartheta)$ is an operator defined the following way:

$$Z(y, z_t(\vartheta), \vartheta) = \begin{cases} -b_k Y(y, z_t(\vartheta)) & (-\tau \leqslant \vartheta < 0) \\ X_k(z_{1t}(\vartheta) + b_1 y, \ldots, z_{nt}(\vartheta) + b_n y] - b_k Y & (\vartheta = 0) \quad (k = 1, \ldots, n) \end{cases} \tag{2.16}$$

We note that $Z(y, z_t(\vartheta), \vartheta)$ is a function of ϑ belonging to L because $f[Z] = 0$. It is obvious that

$$Y(0, 0) \equiv 0, \qquad Z(0, 0, \vartheta) \equiv 0 \tag{2.17}$$

The functional Y and the operator Z satisfy a Lipschitz condition with respect to the variables y and $z_t(\vartheta)$ of the type (1.2) and (1.3).

Since the system (1.8) has a unique solution $x_t(\vartheta)$ under the initial condition $x_0(\vartheta)$ in the neighborhood of the origin $x = 0$, it is obvious that $z_t(\vartheta)$ and $y(t)$ constitute the solution of the svstem (2.14) with the initial conditions $z_0(\vartheta)$ and $y(0)$ if $x_t(\vartheta) = z_t(\vartheta) + by(t)$.

The uniqueness of the solution $z_t(\vartheta)$, $y(t)$ follows from the uniqueness of the decomposition of an arbitrary function $x(\vartheta)$ in terms of $z(\vartheta)$ and yb.

3. Transformation of an equivalent system. (a) Let the symbols $Y^\circ(y)$ and $Z^\circ(y,\vartheta)$ stand for the analytic function

$$Y^\circ(y) \equiv Y(y, 0) = gy^m + \ldots \tag{3.1}$$

$$Z^\circ(y, \vartheta) \equiv Z(y, 0, \vartheta) = g_1(\vartheta)y^{m_1} + \ldots \qquad (g_1(\vartheta) \text{ — vector}, \ m_1 \geqslant 2)$$

It can be shown that there always exists a transformation of the variable z such that $m_1 > m$. The transformation is similar to the transformation used by Liapunov for the case of one zero root of a system described by ordinary equations [3, p. 142]. There is an exceptional (singular) case which will not be considered here.

(b) Let us consider the system of equations

$$Au(\vartheta) + Z(y, u(\vartheta), \vartheta) = 0 \tag{3.2}$$

where $Z \in L$ and is defined by the relagion (2.16) in which $z_t(\vartheta)$ is replaced by $u(\vartheta)$; the operator A is defined by Formula (1.9).

Let us define the operator $A^{-1}(\phi(\vartheta))$ in the following way:

$$A^{-1}(\varphi(\vartheta)) \equiv \left\{-b_k\left(f[x^*] + f\left[\int_k^{\vartheta}\varphi(\vartheta)\,d\vartheta\right]\right) + x_k^* + \int_0^{\vartheta}\varphi_k(\vartheta)\,d\vartheta\right\} \qquad (k = 1, \ldots, n)$$

where $\{x_k^*\} = x^*$ satisfies the system of homogeneous algebraic equations

$$\sum_{j=1}^n a_{kj}x_j^* = \varphi_k(0) + \sum_{l=1}^n \int_0^{-r}\left[\int_k^{\vartheta}\varphi_l(\vartheta_1)\,d\vartheta_1\right]d\eta_{jl}(\vartheta) \quad \left(a_{kj} = \int_{-\tau}^0 d\eta_{kj}(\vartheta)\right) \quad (k = 1, \ldots, n)$$

The last system is solvable if $\phi(\vartheta)$ lies in the L-surface, and, hence, the following condition is satisfied:

$$f[\varphi(\vartheta)] \equiv \sum_{j=1}^n \Delta_{jk_1}(0)\left[-\varphi_j(0) + \sum_{l=1}^n \int_{-\tau}^0\left[\int_0^{\vartheta}\varphi_l(\vartheta_1)\,d\vartheta_1\right]d\eta_{jl}(\vartheta)\right] = 0$$

The operator A^{-1} is bounded (in the norm) on the set of differentiable functions $\phi(\vartheta)$ $(-r < \vartheta < 0)$ belonging to L. The norm of A^{-1} is defined by

$$\| A^{-1}\varphi(\vartheta) \| = \sup(| A_k^{-1}\varphi(\vartheta) | / \| \varphi(\vartheta) \|) = M$$

where M is a positive number.

The vector function $A^{-1}\phi(\vartheta)$ has a derivative with respect to on the interval $[-r, 0]$.

It is not difficult to show that the next identities are valid for any differentiable function $\phi(\vartheta) \in L(-\tau \leqslant \vartheta \leqslant 0)$:

$$AA^{-1}\varphi(\vartheta) = \varphi(\vartheta) \qquad A^{-1}A\varphi(\vartheta) = \varphi(\vartheta) \tag{3.3}$$

Theorem 3.1. The system of Equations (3.2) has a unique solution $u(y, \vartheta)$ analytic in y in the neighborhood of the point $y = 0$, $u(0, \vartheta) = 0$, continuous in ϑ and such that $u(y, \vartheta) \in L$. This solution can be found by satisfying Equation (3.2) with formal power series in terms of positive powers of y with unknown coefficients of the type

$$u(y, \vartheta) = u_{m_1}(\vartheta) y^{m_1} + u_{m_1+1}(\vartheta) y^{m_1+1} + \ldots \tag{3.4}$$

The coefficients are uniquely determined by the condition that $u_j \in L$.

Proof. Because of the properties of A^{-1}, a solution of the equation

$$u(\vartheta) = -A^{-1}Z(y, u(\vartheta), \vartheta) \tag{3.5}$$

will also be a solution of Equation (3.2).

Since the operator A^{-1} is bounded in the norm, the operator Z satisfies the Lipschitz condition with an arbitrarily small coefficient q. Furthermore, the operator A^{-1} transforms every element $\phi(\vartheta) \in L$ into an element $A^{-1}\phi(\vartheta) \in L$. The truth of the theorem now follows from the principle of contraction mappings [9]. The theorem can also be proved without difficulty by the simple method of successive approximations.

(c) Let $u^*(y, \vartheta)$ be a solution of Equation (3.1). We make a change of the variable from $z(\vartheta)$ to $z_1(\vartheta)$ in Equation (2.14):

$$z(\vartheta) = z_1(\vartheta) + u^*(y, \vartheta), \qquad z_1(\vartheta) = \{z_{1k}(\vartheta)\} \tag{3.6}$$

The system (2.14) takes the form

$$\frac{dy}{dt} = Y(y, z_{1t}(\vartheta) + u^*(y, \vartheta)) \tag{3.7}$$

$$\frac{dz_{1t}(\vartheta)}{dt} = Az_{1t}(\vartheta) + Z_1(y, z_{1t}(\vartheta) + u^*(y, \vartheta), \vartheta) \tag{3.8}$$

where

$$Z_1 = Au^*(y, \vartheta) + Z(y, z_1 + u^*(y, \vartheta), \vartheta) - \frac{\partial u^*}{\partial y} Y$$

The corresponding function will be

$$Z_1^\circ(y, \vartheta) \equiv -\frac{\partial u^*(y, \vartheta)}{\partial y} Y(y, u^*(y, \vartheta))$$

Therefore, $m_1 > m$ as long as the function $Y^\circ(y) \equiv Y(y, u^*(y, \vartheta))$ does not vanish identically. The latter case will be called the singular case.

4. Criterion of stability. Let us suppose that the system (1.8) of differential equations with an operator type of right hand side (equivalent to (1.1)) has been reduced with the aid of Liapunov's transformation (Section 3) to the form

$$\frac{dy}{dt} = Y_1(y, z_1 + (\vartheta)), \qquad \frac{dz_1(\vartheta)}{dt} = Az_{1t}(\vartheta) + Z_1(y, z_{1t}(\vartheta), \vartheta) \tag{4.1}$$

Here, $Y_1(y, z_{1t}(\vartheta)) = Y(y, z_{1t}(\vartheta) + u^*(y, \vartheta))$, the function Z_1 is defined in (3.8), and the function $u^*(y, \vartheta)$ is the solution of the system (3.2):

$$Y_1^\circ(y) = Y(y, u^*(y, \vartheta)) = gy^m + \ldots$$
$$Z_1^\circ(y, \vartheta) = -\frac{\partial u^*(y, \vartheta)}{\partial y} Y_1^\circ(y) = g_1 y^{m_1} + \ldots \qquad (m_1 > m)$$

Theorem 4.1. Suppose that the system of equations (1.8) can be reduced by the change of variables (2.11), (3.6) to the form (4.1).

(a) If m is odd and g a negative number then the motion $x = 0$ of the system (1.1) will be asymptotically stable.

(b) If m is even, or if m is odd and g is positive, then the undisturbed motion $x = 0$ will be unstable.

Proof. On the plane L in the function space $x_t(\vartheta)$ one can construct for the linear system

$$\frac{dz_{1t}(\vartheta)}{dt} = Az_{1t}(\vartheta), \qquad f[z_{1t}(\vartheta)] = 0 \tag{4.2}$$

a functional $v_2(z_{it}(\vartheta), t)$ satisfying the following conditions:

$$c_1 \| z_{1t}(\vartheta) \| \leqslant v_2(z_{1t}(\vartheta), t)) \leqslant c_2 \| z_{1t}(\vartheta) \| \tag{4.3}$$

$$\lim\sup \left(\frac{\Delta v_2}{\Delta t}\right) \leqslant -c_3 \| z_{1t}(\vartheta) \| \quad \text{при } \Delta t \to +0 \tag{4.4}$$

$$| v_2(z_{1t}''(\vartheta), t) - v_2(z_{1t}'(\vartheta), t) \leqslant c_4 \| z_{1t}'' - z_{1t}' \| \tag{4.5}$$

(This follows from the results of [1, pp. 191-192].

Here c_1, c_2, c_3, and c_4 are positive constants.

We construct a functional $v^*(x_t(\vartheta), t)$ for the system (1.8) (or (4.1)) of the form

$$v^*(x_t(\vartheta), t) = -gy^{m+1} + v_2{}^2(z_{1t}|\vartheta|, t) \tag{4.6}$$

where y and $z_{1t}(\vartheta)$ are determined by means of $x_t(v)$ according to Formulas (2.11), (2.14) and (3.6).

Suppose $g < 0$, and m is odd. One finds positive constants B, C, D such that the following inequalities are satisfied:

$$\|y\| < B\|x_t(\vartheta)\|, \quad \|z_{1t}(\vartheta)\| < C\|x_t(\vartheta)\|, \quad \|x_t(\vartheta)\| < D\{\|y\| + \|z_{1t}(\vartheta)\|\} \tag{4.7}$$

By making an upper estimate of $v^*(x_t(\vartheta), t)$ we find that in the region $\| x_t(\vartheta) \| < H$

$$\|v^*(x_t(\vartheta), t)\| < (|g|B^{m+1}H^m + Hc_2{}^2)\|x_t(\vartheta)\| \tag{4.8}$$

On the set $r \leqslant \| x_t(\vartheta) \| \leqslant H$, the functional $v^*(x_t(\vartheta), t)$ cannot take on arbitrarily small values. This functional is bounded from below by a positive number. Indeed, let us assume that $\| v^* \|$ (for some function $x^*(\vartheta)$ satisfying the condition $r \leqslant \| x^*(\vartheta)\| \leqslant H$) is less than an arbitrarily small positive number δ. Then

$$\|x^*(\vartheta)\| < D(\|y^*\| + \|z_1{}^*(\vartheta)\|) < D\left(|g|^{-1}\delta^{\frac{1}{m+1}} + c_1{}^{-1}\delta^{\frac{1}{2}}\right)$$

The last inequality contradicts the condition $\| x^*(\vartheta) \| > r$.

Let $f(r)$ stand for the exact lower boundary of the functional v^* on the set $r < \| x(\vartheta) \| < H$. The function $f(r)$ is monotone decreasing if $r_1 > r_2$, then $f(r_1) > f(r_2)$); furthermore, $f(r) < K\| x_t(\vartheta) \|$ because of (4.8).

It is well known that the function $f(r)$ can be represented, on $0 \leqslant r \leqslant H$, as the sum of two positive functions $v(r)$ and $s(r)$, where $v(r)$ is a continuous increasing function, while $s(r)$ is a function with jumps [8].

Hence, if $\| x(\ ) \| < H$, we have that

$$\begin{aligned} v^*(x_t(\vartheta), t) &= |g| \cdot \|y(x_t(\vartheta)\|^{m+1} + v_2{}^2(z_{1t}(\vartheta), t) \geqslant \\ &\geqslant |g| \cdot \|y(x_t(\vartheta))\|^{m+1} + c_1{}^2\|z_{1t}(v)\|^2 \geqslant w(\|x_t(\vartheta)\|) \end{aligned} \tag{4.9}$$

Along a trajectory of the system (1.8) we find that

$$\limsup_{\Delta t\to+0}\left(\frac{\Delta v}{\Delta t}\right)^*_{(1.8)} = -g^2(m+1)y^{2m} + 2|v_2|\limsup_{\Delta t\to+0}\left(\frac{\Delta v_2}{\Delta t}\right)_{(4.2)} -$$
$$-g(m+1)y^mY_1(y, z_{1t}(\vartheta)) + 2|v_2|\limsup_{\Delta t\to+0}\left[\left(\frac{\Delta v_2}{\Delta t}\right)_{(1.8)} - \left(\frac{\Delta v_2}{\Delta t}\right)_{(4.2)}\right] \leqslant$$
$$\leqslant -g^2(m+1)y^{2m} - 2c_1c_2\|z_{1t}(\vartheta)\|^2 + g(m+1)y^m[|Y_1(y,0)| +$$
$$+ q\|z_{1t}(\vartheta)\|] + 2c_2c_4\|z_{1t}(\vartheta)\|(\|Z_1(y,0)\| + q\|z_{1t}(\vartheta)\|)$$

Let us select H_1 so small that the following inequalities are satisfied:

$$g^2 - 3gY_1(y,0)y^{-m} > 0$$
$$2c_1c_2 - 3c_2c_4q(\|y\|\|z_{1t}(\vartheta)\|) > 0$$
$$-g^2(m+1)y^{2m} - 2c_1c_3\|z_{1t}(\vartheta)\| + 3q\|z_{1t}(\vartheta)\|y^m < 0$$

The last inequality will hold if

$$q < \frac{\sqrt{8}}{3}g\sqrt{m+1}\cdot\sqrt{c_1c_3}$$

Thus, we obtain the following estimate:

$$\limsup_{\Delta t\to+0}\left(\frac{\Delta v^*}{\Delta t}\right)_{(1.18)} \leqslant \frac{1}{3}[g^2(m+1)y^{2m} + 2c_1c_3\|z_{1t}(\vartheta)\|^2]$$

One can show in a manner analogous to the one used for estimating v^*, that there exists a continuous monotone increasing positive function w_1 such that $(w_1(r)r > 0,\ w_1(0) = 0)$

$$\limsup_{\Delta t\to\infty}\left(\frac{\Delta v^*}{\Delta t}\right)_{(1.8)} \leqslant -w_1(r) \tag{4.10}$$

where $w_1(r)$ is found in a way similar to the way in which $w(r)$ was found in Formula (4.9).

From conditions (4.8)-(4.10) it now follows, in accordance with Theorem 30.1 of [5, p. 194], that the motion $x = 0$ of the system (1.8) and (1.1) will be asymptotically stable if $g < 0$, and m is an odd number.

Next, suppose that $g > 0$ and that m is odd (or m is even). For the functional v^* for the system (1.8) we select the functional

$$v^*(x_t(\vartheta), t) = gy^{m+1} - v_2^2(z_{1t}(\vartheta), t) \tag{4.11}$$

We compute the following limits:

$$- \lim_{\Delta t \to \infty} \inf \left(\frac{\Delta v^*}{\Delta t}\right)_{(1.8)} = -(m+1) g^2 y^{2m} - (m+1) g y^m Y_1 (y, z_{1t} (\vartheta)) -$$
$$- \lim_{\Delta t \to \infty} \inf \left(\frac{\Delta (-v_2^2)}{\Delta t}\right)_{(4.2)} - \lim_{\Delta t \to +0} \inf \left[\left(\frac{\Delta (-v_2^2)}{\Delta t}\right)_{(1.8)} - \left(\frac{\Delta (-v_2^2)}{\Delta t}\right)_{(4.2)}\right] =$$
$$= -(m+1) g^2 y^{2m} + \lim_{\Delta t \to 0} \sup \left(\frac{\Delta v_2^2}{\Delta t}\right)_{(4.2)} +$$
$$+ (m+1) g y^m Y_1 (y, z_t (\vartheta)) + \lim_{\Delta t \to 0} \sup \left[\left(\frac{\Delta v_2^2}{\Delta t}\right)_{(1.8)}\right] - \lim_{\Delta t \to +0} \sup \left(\frac{\Delta v_2^2}{\Delta t}\right)_{(4.2)}$$

Making estimates analogous to those made in the derivativation of Formula (4.10), we obtain (if $\| x_t(\vartheta) \|$ is small enough and $\| x_t(\vartheta) \| \leqslant H_1 < H$)

$$\lim_{\Delta t \to \infty} \inf \left(\frac{\Delta v}{\Delta t}\right)^*_{(1.8)} \geqslant w_3 (\| x_t (\vartheta) \|) \tag{4.12}$$

where the formula w_3 is analogous to $w(r)$, and $w_3(r) r > 0$, $w_3(0) = 0$.

The left-hand side of (4.12) is thus of definite sign and admits an arbitrarily small upper bound. One can easily show that v^* is bounded in the region $\| x_t(\vartheta) \| \leqslant H_1$.

We shall show that by the proper choice of $x_0(v)$ (arbitrarily small in norm) the functional v^* can be made positive. Indeed, let us select

$$y(0) = \eta \neq 0, \qquad z_{10}(\vartheta) = 0 \qquad (z_0(\vartheta) = u^*(\eta, \vartheta))$$

In accordance with (2.11) and (3.6) the corresponding expression $x_0{}^*(\vartheta)$ will be

$$b\eta + u^*(\eta, \vartheta)$$

Substituting $x_0{}^*$ into the function v^*, one can verify that $v^*(x_0{}^*(\vartheta)) > 0$ if the norm of $\| x_0{}^*(\vartheta) \|$ is small enough.

Here the sign of v^* will be positive for all $x_0{}^*(\vartheta)$ with small norms. Thus, all the conditions of Liapunov's first theorem on the instability of motion [10] are satisfied, and that theorem applies, therefore, also to systems with lag. The motion $x = 0$ of the systems (1.8) and (1.1) is unstable.

Note 4.1. From the established theorem (4.1) it follows that the stability or instability of the motion $x = 0$ are determined by the sign of g or by the form of the lowest degree term of the function $Y^0(y) = gy^m + \ldots$ Hence, in the reduction of the system (2.14) to the form (4.1) it is sufficient to take, in place of the solution of the system (3.2), the approximate solution satisfying the system (3.2) with a degree of approximation of up to a term of degree $(m+1)$ in y.

Note 4.2. If $Y_1(y, 0) \equiv Y(y, u(y, \vartheta)) \equiv 0$, we have the singular case. The motion $x = 0$ will be stable in this case.

5. *Example 1.* Let us consider the following system with lag:

$$\begin{aligned} \frac{dx_1(t)}{dt} &= x_2(t) + X_1(x_1(t), x_2(t), x_1(t-\tau), x_2(t-\tau)) \\ \frac{dx_2(t)}{dt} &= ax_2(t-\tau) + X_2(x_1(t), x_2(t), x_1(t-\tau), x_2(t-\tau)) \end{aligned} \tag{5.1}$$

where X_1 and X_2 are analytic functions whose power-series expansions around the point $x = 0$ begin with the second-degree terms.

The system of the first approximation is

$$\frac{dx_1(t)}{dt} = x_2(t), \qquad \frac{dx_2(t)}{dt} = ax_2(t-\tau) \tag{5.2}$$

Its characteristic equation

$$\Delta(\lambda) \equiv \lambda^2 - \lambda a e^{-\lambda\tau} = 0 \tag{5.3}$$

has one zero root. The remaining roots will have negative real parts if the parameter a satisfies the following condition

$$-\pi/2\tau < a < 0 \tag{5.4}$$

The operator $A[x_1, x_2]$ has the form

$$y_i(\vartheta) = \frac{dx_i(\vartheta)}{d\vartheta} \qquad (-\tau \leqslant \nu < 0) \quad (i=1,2) \qquad y_1(0) = x_2(0), \qquad y_2(0) = ax_2(-\tau)$$

The operator $R(x_1, x_2)$ will be

$$\mathrm{R}(x_1, x_2) = \begin{cases} 0 & (-\tau \leqslant \vartheta < 0), \\ X_i(x_{1t}(\vartheta), x_{2t}(\vartheta)) & (\vartheta, =0), \ x_{it}(\vartheta) = x_i(t+\vartheta) \quad (i=1,2) \end{cases}$$

The functional (2.3) for the system (5.2) will have the form

$$f[x_1, x_2] \equiv ax_{1t}(0) - x_{2t}(0) - a\int_0^{-\tau} x_{2t}(\vartheta)\,d\vartheta, \qquad b = (b_1, b_2) \equiv (a^{-1}, 0) \tag{5.5}$$

The transformation (2.11) becomes

$$y = ax_{1t}(0) - x_{2t}(0) + a\int_0^{-\tau} x_{2t}(\vartheta)\,d\vartheta, \; x_{1t}(\vartheta) = z_{1t}(\vartheta) + \frac{1}{a}y, \quad x_{2t}(\vartheta) = z_{2t}(\vartheta) \tag{5.6}$$

The system is hereby transformed to the form (2.14) in which

$$Y\,(y, z_t\,(\vartheta)) \equiv aX_1\,(x_t(\vartheta)) - X_2\,(x_t\,(\vartheta))$$

$$Z_1 = \begin{cases} -\frac{1}{a}\,Y \;\; (-t \leqslant \vartheta < 0), \\ X_1 - \frac{1}{a}\,Y \quad (\vartheta = 0), \end{cases} \qquad Z_2 = \begin{cases} 0 & (-\tau \leqslant \vartheta < 0) \\ X_2 & (\vartheta = 0) \end{cases} \tag{5.7}$$

where $x_{it}(\vartheta)$ is determined by (1.2). Let us suppose that

$$X_1 \equiv k_1 x_{1t}{}^2\,(0) + k_2 x_{2t}{}^2\,(0), \qquad X_2 \equiv l_1 x_{1t}{}^2\,(-\tau) + l_2 x_{1t}\,(0)\,x_{2t}\,(0)$$

where k_1, k_2, l_1 and l_2 are constants. Equations (3.6) become in this case

$$\begin{gathered} \frac{dz_{1t}\,(\vartheta)}{d\vartheta} - \frac{1}{a}\,Y = 0, \qquad \frac{dz_2}{d\vartheta} = 0 \qquad (-\tau \leqslant \vartheta < 0) \\ z_2\,(0) + \frac{1}{a}\,X_2 = 0, \qquad z_2\,(-\tau) + \frac{1}{a}\,X_2 = 0 \qquad (\vartheta = 0) \end{gathered} \tag{5.8}$$

We shall look for a solution of (5.8) in the form

$$z_1 = v_2\,(\vartheta)\,y^2 + v_3\,(\vartheta)\,y^3 + \cdots, \qquad z_2 = u_2\,(\vartheta)\,y^2 + u_3\,(\vartheta)\,y^3 + \cdots \tag{5.9}$$

where v_i, u_i are unknown coefficients. Substituting (5.9) and (5.8) and equating coefficients of like powers of y, we obtain

$$\frac{dv_2}{dv} = a^{-3}\,(k_1 a - l_1), \qquad u_2 = -\,l_1 a^{-3}$$

We can find the function v_2 from this equation and from the condition that the point $(u_1,\ u_2)$ belongs to L, i.e. from the fact that $f[\ v_2,\ u_2\ ] = 0$. We obtain

$$v_2 = \frac{1}{a^3}(ak_1 - l_1)\,\vartheta - \frac{1}{a^4}\,l_1\,(1 - a\tau)$$

Let us consider the expression

$$Y^\circ\,(y, z\,(y)) = (ak_1 - l_1)\,a^{-2}y^2 + y^3\,(\,\ldots\,) +$$

When $ak_1 - l_1 \neq 0$ we have unstable motion. If $ak_1 - l_1 = 0$ one has to consider the coefficient of y^3 in Y°

$$Y^{(0)}\,(y, z\,(y)) = \frac{l_1 l_2}{a^4}\,y^3 + y^4\,(\,\ldots\,) + \cdots \tag{5.10}$$

If $l_1 l_2 < 0$, one has asymptotic stability, and if $l_1 l_2 > 0$, the motion $x = 0$ is unstable.

Let us suppose that $l_1 l_2 = 0$. One must now consider the coefficient of the fourth power of y in the expression for Y°. For this purpose one has to find u_3 and v_3 in the expansion (5.9) when $l_2 = 0$ and when $l_1 = 0$. In the first case, the coefficient of y_4 in (5.10) will be equal to $-k_2 l_1 a^{-6}$. In this case, if $l_1 k_2 \neq 0$, one has unstable motion. If $l_2 = k_2 = 0$, the singular case occurs. In the second case, when $l_1 = k_1 = 0$, we have the singular case, and hence stability for the motion $x = 0$.

We thus see that all possible cases can be reduced to the five types

(a) $ak_1 - l_1 \neq 0$ (the undisturbed motion is unstable)

(b) $ak_1 - l_1 = 0$, $l_1 l_2 > 0$ (the undisturbed motion is asymptotically stable)

(c) $ak_1 - l_1 = 0$, $l_1 l_2 < 0$ (the undisturbed motion is unstable)

(d) $ak_1 - l_1 = 0$, $l_2 = 0$, $l_1 \neq 0$ (the undisturbed motion is unstable)

(e) $ak_1 - l_1 = 0$, $l_2 = 0$, $k_2 = 0$ or $l_1 = 0$, $k_1 = 0$ (the undisturbed motion is stable; one has the singular case)

Example 2. Let us consider

$$\frac{d^2x\,(t)}{dt^2} = a\,\frac{dx\,(t-\tau)}{d\tau} + X\,(x\,(t),\ x^{\cdot}\,(t),\ x\,(t-\tau),\ x^{\cdot}\,(t-\tau)) \tag{5.11}$$

where X is an analytic function of its arguments and a is a constant.

With the aid of the substitution $x_1(t) = x(t)$, $x_2(t) = x(t)$, the system (5.11) can be reduced to the system (5.1) in which $X_1 \equiv 0$, $X_2 \equiv X$. Since the lowest degree terms in y in the functions Y° and Z° are always the same in this case, we obtain the following results.

We consider the function

$$X\left(\frac{1}{a}\,y,\ 0,\ \frac{1}{a}\,y,\ 0\right) = gy^m + \ldots$$

If m is odd and $g < 0$, we have asymptotically stable motion. If m is odd and $g > 0$, or if m is even, we have instability of the undisturbed motion. Finally, if

$$X\left(\frac{1}{}\,y, 0, \frac{1}{a}\,y,\ 0\right) \equiv 0$$

we have stability of the undisturbed motion (the singular case).

BIBLIOGRAPHY

1. Krasovskii, N.N., *Nekotorye zadachi teorii ustoichivosti dvizheniia* (*Some Problems of the Theory of the Stability of Motion*). Fizmatgiz, 1959.

2. Riss, F. and Sekefal'vi-Nad'B., *Lektsii po funktsional'nomu analizu (Lectures on the Functional Analysis)*. Moscow, 1954.

3. Liapunov, A.M., *Obshchaia zadacha ob ustoichivosti dvizheniia (General Problem on the Stability of Motion)*. Gostekhizdat, 1950.

4. Chetaev, N.G., *Ustoichivosti dvizheniia (Stability of Motion)*. Gostekhizdat, 1955.

5. Malkin, I.G., *Teoriia ustoichivosti dvizheniia (Theory of Stability of Motion)*. Gostekhizdat, 1952.

6. Kamenkov, G.V., Ob ustoichivosti dvizheniia (On the stability of motion). *Sb. tr. Kazanskogo aviats. in-ta* No. 9, 1939.

7. Bellman, R., On the existence and boundedness of solutions of non-linear differential-difference equations. *Ann. of Math.* Vol. 50, No. 2, 1959.

8. Natanzon, I.P., *Teoriia funktsii veshchestvennoi peremennoi (Theory of Functions of a Real Variable)*. Gostekhizdat, 1950.

9. Krasnosel'skii, M.A., *Topologicheskie metody v teorii nelineinykh integral'nykh uravnenii (Topological Methods in the Theory of Non-linear Integral Equations)*. Gostekhizdat, 1956.

10. Shimanov, S.N., O neustoichivosti dvizheniia sistem s zapazdyvaniem (On the instability of the motion of systems with delay). *PMM* Vol. 23, No. 1, 1960.

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